ON ALMOST HOLOMORPHIC LAGRANGIAN FIBRATIONS

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Abstract. We prove that the pull back of an ample line bundle by an almost holomorphic Lagrangian fibration is nef. As an application, we show birational semi rigidity of Lagrangian fibrations.

1. Introduction

We start with the definition of the almost holomorphic fibration.

Definition 1.1. *A rational map f is said to be almost holomorphic if there exists an open set U on which f is defined and the induced morphism $U \to S$ is proper.*

We recall the definition of the pull back of a Cartier divisor by a rational map.

Definition 1.2. *Let $f : X \dashrightarrow S$ be a rational map and D_S a Cartier divisor on S. We define the pull back f^*D_S by $\nu_* g^* D_S$, where $\nu : Y \to X$ is a resolution of the indeterminacy of f and g is the induced morphism $Y \to S$. Note that this definition does not depend on the choice of resolution because every push-forwards of exceptional divisors are zero. We could consult* [Nak04, §2.e] *the fundamental facts of the push-forwards of divisors.*

Theorem 1.1. *Let $f : X \dashrightarrow S$ be an almost holomorphic fibration from a projective symplectic manifold. Assume that a general fibre of f is a Lagrangian submanifold. Let D_S be an ample divisor of S. Then the pull back f^*D_S is nef.*

Remark 1.1. *In the case that X is a non projective irreducible symplectic manifold, the same assertion holds by* [COP10, Theorem 1.2].

Remark 1.2. *Professors Daniel Greb, Christian Lehn and Sönke Rollenske pointed out that Theorem 1.1 holds without the assumption that a general fibre is Lagrangian if X is an irreducible symplectic manifold. Indeed, by Proposition 2.1, there exists a projective irreducible symplectic manifold X_1 and a birational map $\phi : X \dashrightarrow X_1$ such that $\phi_*(f^*D_S)$ is nef. Since ϕ is isomorphic in a neighbourhood of a general fibre of $f : X \dashrightarrow S$, X_1 admits also an almost holomorphic fibration and $\phi_*(f^*D_S)$ is trivial on a general fibre. Thus the nef dimension of $\phi_*(f^*D_S)$ is strictly less than $\dim X_1$. By* [Mat08, Theorem 1.1], *$\phi_*(f^*D_S)$ is semiample and induces a Lagrangian fibration $f_1 : X_1 \to S_1$. A general fibre of f is the complete intersection of general members of the linear system of $|f^*D_S|$ and a general fibre of f_1 is also the complete intersection of general members of the linear system of $|\phi_*(f^*D_S)|$. Thus a general fibres of f and f_1 coincide. This implies that a general fibre of f is Lagrangian.*

The following is a geometric interpretation of Theorem 1.1.

Corollary 1.1. *Under the same assumption of Theorem 1.1, we let U be the maximal open set of X on which f is defined. Then there exists a Lagrangian fibration*

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$g : X \to T$ which satisfies the following diagram:

$$\begin{tikzcd} X \arrow[d, "g"'] & U \arrow[l, hook'] \arrow[d, "f"] \\ T & f(U) \arrow[l, hook'] \end{tikzcd}$$

Remark 1.3. *There exists an example of an almost holomorphic Lagrangian fibration which is not holomorphic. Let $f : X \to \mathbb{P}^2$ be a Lagrangian fibration. The composition of f and a Cremona transformation of $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ gives such an example.*

Remark 1.4. *Let X be an irreducible symplectic manifold which contains a Lagrangian submanifold A isomorphic to a complex torus. In* [Bea11, Question 1.6]*, it is asked whether X admits a Lagrangian fibration under this condition. In* [GLR11a, Theorem 4.1]*, they proved that X admits a Lagrangian fibration if X is not projective. In the case that X is projective, it is proved that X admits an almost holomorphic fibration whose general fibre is a Lagrangian submanifold by* [Ame11, Theorem 3] *if* $\dim X = 4$ *and* [HW12, Theorem 1.2] *in any dimension. In the case* $\dim X = 4$*, they proved that X admits a Lagrangian fibration in* [GLR11b, Theorem 5.1]*. Together with Corollary 1.1, X admits a Lagrangian fibration in all cases.*

As an application, we obtain semi birational rigirity of Lagrangian fibrations.

Corollary 1.2. *Let $f_1 : X_1 \to S_1$ and $f_2 : X_2 \to S_2$ be Lagrangian fibrations from projective symplectic manifolds. Assume that they satisfy the following diagram*

$$\begin{tikzcd} X_1 \arrow[d] & X_2 \arrow[l, dashed, "\phi_1"'] \arrow[d] \\ S_1 & S_2 \arrow[l, dashed, "\phi_2"] \end{tikzcd}$$

where ϕ_1 and ϕ_2 are birational maps. Then ϕ_2 is isomorphic.

Example 1.1. *Let A be an abelian surface with an elliptic fibration $\varphi : A \to F$. The generalized Kummer variety $\mathrm{Km}^{n+1}A$ admits a Lagrangian fibration $\mathrm{Km}^{n+1}A \to \mathbb{P}^n$. Let us consider the intersection of $\mathrm{Km}^{n+1}A$ and the relative Hilbert scheme $\mathrm{Hilb}^{n+1}(A/F)$ in $\mathrm{Hilb}^{n+1}A$. We denote by f a torsion point of F of order $n+1$ and by E the fibre of φ at f. We define the map*

$$h_f : \mathrm{Hilb}^{n+1}E \to \mathrm{Ker}(\varphi),$$

by sending a zero cycle of E into its sum of A. By [OS11, Proposition 6.4]*, $P := \mathrm{Hilb}^{n+1}(A/F) \cap \mathrm{Km}^{n+1}A$ consists of the disjoint union of the preimage $h_f^{-1}(0)$. Hence P is the disjoint union of 2^{n+1} copies $\mathbb{P}^n$ and is contained in fibres of the Lagrangian fibration. The Mukai flop $(\mathrm{Km}^{n+1}A)'$ along P admits a Lagrangian fibration and we have the following diagram:*

$$\begin{tikzcd} \mathrm{Km}^{n+1}A \arrow[r, dashed, "\phi"] \arrow[d] & (\mathrm{Km}^{n+1}A)' \arrow[d] \\ \mathbb{P}^n \arrow[r, equal] & \mathbb{P}^n \end{tikzcd}$$

where ϕ is not isomorphic.

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2. Proof of Theorem 1.1

Proof of Theorem 1.1. We start with proving the following Proposition.

Proposition 2.1. *Let X be a projective symplectic manifold and D an effective divisor on X. Assume that the linear system $|D|$ associated with D has no fixed divisor. Then there exists a birational map $\phi : X \dashrightarrow X'$ and a projective symplectic manifold X' such that $\phi_* D$ is nef and ϕ is isomorphic on the open set $X \setminus \mathrm{Bs}|D|$, where $\mathrm{Bs}|D|$ stands for the base locus of the linear system $|D|$.*

Proof. If D is nef, there are nothing to proof. Assume that D is not nef. We choose a small rational number δ such that the pair $(X, \delta D)$ has only Kawamata log terminal singularities. By [KMM87, Theorem 3-2-1], there exists an extremal ray of the pair $(X, \delta D)$ and we have a contraction morphism $\pi_0^- : X \to Z_0$ such that $-D$ is π_0^--ample by [KMM87, Theorem 4-2-1]. If we choose an another effective ample divisor D' which is linearly equivalent to D, $-D'$ is also π_0^--ample. Hence the exceptional locus of π_0^- is contained in the base locus of $|D|$. Since $|D|$ has no fixed component, π_0^- is small. By [BCHM10, Corollary 1.4.1], we have a birational morphism $\pi_0^+ : X_1 \to Z_0$ such that $(\phi_1)_* D$ is π_0^+-ample, where ϕ_1 is the induced birational map $X \dashrightarrow X_1$. If $(\phi_1)_* D$ is not nef, we repeat this process and obtain the following diagram:

$$\begin{tikzcd}
X \arrow[rr, dashed, "\phi_1"] \arrow[dr, "\pi_0^-"'] & & X_1 \arrow[rr, dashed, "\phi_2"] \arrow[dl, "\pi_0^+"] \arrow[dr, "\pi_1^-"'] & & X_2 \arrow[rr, dashed, "\phi_3"] \arrow[dl, "\pi_1^+"] \arrow[dr, "\pi_2^-"'] & & X_3 \arrow[dl, "\pi_2^+"] & \cdots & & X_n \arrow[dl, "\pi_{n-1}^+"] \\
& Z_0 & & Z_1 & & Z_2 & \cdots & & Z_{n-1} &
\end{tikzcd}$$

We denote $(\phi_i)_* D_{i-1}$ by D_i, where $D_0 := D$. We also denote by $\phi_i : X_{i-1} \dashrightarrow X_i$ the induced birational maps. Since π_i^- is $-D_i$ ample, the exceptional locus of π_i^- is contained in the base locus of $|D_i|$ and ϕ_i is isomorphic over $X_{i-1} \setminus \mathrm{Bs}|D_{i-1}|$. Let U_i be the image $X_{i-1} \setminus \mathrm{Bs}|D_{i-1}|$ by ϕ_i. Since $\mathrm{Bs}|D_i| \cap U_i = \emptyset$, ϕ_{i+1} is defined and isomorphic over U_i. Hence the composition maps $\phi_n \circ \cdots \circ \phi_1$ are defined and isomorphic over $U_0 = X \setminus \mathrm{Bs}|D|$ for all n. The following Lemma completes the proof of Proposition.

Lemma 2.1. (1) *All X_i are projective symplectic manifolds*
(2) *After finite steps, D_n becomes nef.*

Proof. The assertion (1) follows by [Nam06, Corollary 1]. Assume that D_n is not nef for all n. Then we have an infinite chain of D-flops. By [Mat09, Theorem 1.2], there exist no such chains. For reader's convenience, we give a brief explanation. According to [Sho04, Theorem], such a chain does not exist if the following two conditions are satisfied:

(1) The function $p_i(x)$ on X_i defined by
$$X_i \ni x \mapsto p_i(x) := \mathrm{mld}(x; X_i, \delta D_i)$$
is lower semicontinuous.
(2) Let W_i be the exceptional locus of π_i^-. The set $\bigcup_i \mathrm{mld}(W_i; X_i, \delta D_i)$ satisfies the ascending chain condition.

We could find the definition of $\mathrm{mld}(x; X, D)$ in [EMY03, Definition 1.1]. Since all X_i are smooth, the condition (1) is satisfied by [EMY03, Theorem 4.4]. Moreover if we choose an integer N for that $N\delta$ is an integer, all discrepancies are integers after N-multiple. By [EMY03, Definition 1.1], minimal discrepancies are less than $\dim X + 1$. Hence the condition (2) is also satisfied. That is a contradiction. $\square$

We complete the proof of Proposition 2.1. $\square$

Let us start the proof of Theorem 1.1. If f^*D_S is nef, there are nothing to prove. Assume that f^*D_S is not nef. By Proposition 2.1, for the pair (X, f^*D_S), we have the following diagram:

$$\begin{tikzcd}
X \arrow[rr, dashed, "\phi_1"] \arrow[dr, "\pi_0^-"'] & & X_1 \arrow[rr, dashed, "\phi_2"] \arrow[dl, "\pi_0^+"] \arrow[dr, "\pi_1^-"'] & & X_2 \arrow[rr, dashed, "\phi_3"] \arrow[dl, "\pi_1^+"] \arrow[dr, "\pi_2^-"'] & & X_3 \arrow[dl, "\pi_2^+"] & \cdots & X_n \arrow[dl, "\pi_{n-1}^+"] \\
& Z_0 & & Z_1 & & Z_2 & \cdots & Z_{n-1} &
\end{tikzcd}$$

where X_i are symplectic manifolds and the proper transform $(\phi_n \circ \cdots \circ \phi_1)_*(f^*D_S)$ is nef. We denote by D_i the proper transform of f^*D_S by the composition map $\phi_i \circ \cdots \circ \phi_1$. Then D_i is π_{i-1}^+ ample.

Lemma 2.2. *The proper transform D_n of f^*D_S on X_n is semiample and induces a Lagrangian fibration $f_n : X_n \to S_m$.*

Proof. By Proposition 2.1, $\phi_n \circ \cdots \circ \phi_1$ is isomorphic in a neighbourhood of a general fibre of $f : X \dashrightarrow S$. Hence X_n also admits an almost holomorphic Lagrangian fibration $g : X_n \dashrightarrow S_n$. Since $\kappa(D_n) = \kappa(D_S)$, D_n is nef and good. By [Fuj11, Theorem1.1], D_n is semiample. Let $f_n : X_n \to S_n$ be the induced morphism. Since D_n is trivial on a general fibre of g, f_n is a Lagrangian fibration. $\square$

We go back to the proof of Theorem 1.1. Let E be an irreducible component of the exceptional locus of π_{n-1}. We denote by F a general fibre of the induced morphism $f_n|_E : E \to f_n(E)$. We note that $f_n(E) \neq S_n$ because the composition map $\phi_n \circ \cdots \circ \phi_1 : X \dashrightarrow X_n$ is isomorphic in a neighbourhood of a general fibre of the original almost holomorphic fibration $X \dashrightarrow S$ by Proposition 2.1. We take an embedded resolution $\nu : Y \to X_n$ of F. We denote by $\tilde{F}$ the proper transform of F and by p the image of F. The following diagram shows the relationship of the defined objects:

$$\begin{tikzcd}
& Y \arrow[d, "\nu"] & & \tilde{F} \arrow[ll, hook'] \arrow[d] \\
Z_{n-1} & X_n \arrow[l, "\pi_{n-1}^+"'] \arrow[d, "f_n"] & E \arrow[l, hook'] \arrow[d, "f_n"] & F \arrow[l, hook'] \arrow[d] \\
& S_n & f_n(E) \arrow[l, hook'] & \{p\} \arrow[l, hook']
\end{tikzcd}$$

We will prove the following two Claims which contradicts each other.

Claim 2.1. *Let ω be a symplectic form on X_k whose restriction to a gereral fibre of f_n is zero. Then $\nu^*\omega|_{\tilde{F}}$ is zero.*

Proof. We consider the following morphism:

$$R^2(f_n \circ \nu)_*\mathcal{O}_Y \otimes k(p) \to H^2(Y_p, \mathcal{O}) \to H^2(\tilde{F}, \mathcal{O})$$

where $k(p)$ stands for the residue field at p and Y_p the fibre of $f_n \circ \nu$ at p. By the assumption the restriction of $\bar{\omega}$ to the restriction to the general fibre of $f_n \circ \nu$ is zero. By [Kol86, Theorem 2.1], $R^2(f_n \circ \nu)_*\mathcal{O}_Y$ is torsion free. Hence the image of

$\nu^*\bar{\omega}$ in $R^2(f_n \circ \nu)_*\mathcal{O}_Y \otimes k(p)$ is zero. Therefore the image of $\nu^*\bar{\omega}$ in $H^2(\tilde{F}, \mathcal{O})$ is also zero. This implies that $\nu^*\omega|_{\tilde{F}} = 0$. □

Claim 2.2. *Under the same notation as Claim 2.1, $\nu^*\omega|_F$ is not zero.*

Proof. We recall two geometric natures of the image of a birational contraction from a symplectic manifold. Let $\pi^+_{n-1}(E)^\circ$ be the smooth locus of the image of E and ω a symplectic form on X_n. By [Kal06, Lemma 2.9], $\pi^+_{n-1}(E)^\circ$ carries a symplectic form ω_E such that $\nu^*\omega_E = \omega$ on E. Moreover

$$\dim \pi^+_{n-1}(E) = \dim X_n - 2(\dim X_n - \dim E) = 2\dim E - \dim X_n$$

by [Wie03, Theorem 1.2 (ii)]. We estimate the dimension of $\pi^+_{n-1}(F)$. Since $f_n(E) \neq S_n$, $\dim F \geq \dim E - (1/2)\dim X_n + 1$. Let C be a contracted curve by π^+_{n-1}. Since D_n is π^+_{n-1}-ample, $D_n.C > 0$. Hence C is not contracted by f_n and all curves contained in F is not contracted by π^+_{n-1}. We have

$$\dim \pi^+_{n-1}(F) \geq \dim E - \frac{1}{2}\dim X_n + 1.$$

This implies $2\dim \pi^+_{n-1}(F) > \dim \pi^+_{n-1}(E)$ and we are done. □

We complete the proof of Theorem 1.1. □

Proof of Corollary 1.2. Let Y be the common resolution of X_1 and X_2. We denote by ν_i, $(i = 1, 2)$ the birational morphisms $Y \to X_i$.

Claim 2.3. *Let D_{S_1} be an ample divisor on S_1. There exists a $\mathbb{Q}$-Cartier divisor D_{S_2} such that*

$$(f_1 \circ \nu_1)^* D_{S_1} \sim (f_2 \circ \nu_2)^* D_{S_2}$$

Proof. By Theorem 1.1, $(\nu_2)_*(f_1 \circ \nu_1)^* D_{S_1}$ is nef. By [Mat00, Theorem 1], f_2 is equidimensional. Since X_2 is smooth, S_2 is $\mathbb{Q}$-factorial. By the definition, $(\nu_2)_*(f_1 \circ \nu_1)^* D_{S_1}$ does not contain a general fibre of f_2. Thus $f_2((\nu_2)_*(f_1 \circ \nu_1)^* D_{S_1})$ defines a divisor of S_2. We define

$$D_{S_2} := \min\{\Delta | \text{a } \mathbb{Q}\text{-divisor on } S_2 \text{ such that } f_2^*\Delta - (\nu_2)_*(f_1 \circ \nu_1)^* D_{S_1} \geq 0\}.$$

If $f_2^* D_{S_2} \neq (\nu_2)_*(f_1 \circ \nu_1)^* D_{S_1}$, then $H := -f_2^* D_{S_2} + (\nu_2)_*(f_1 \circ \nu_1)^* D_{S_1}$ is f_2-nef and $-H$ is effective. That is a contradiction. □

We go back to the proof of Corollary 1.2. Let C be a curve contained in a fibre of $f_2 \circ \nu_2$. Then C is contracted by $f_1 \circ \nu_1$ because $(f_1 \circ \nu_1)(C).D_{S_1} = 0$ by Claim 2.3. By the same argument, a curve contained in a fibre of $f_1 \circ \nu_1$ is contracted by $f_2 \circ \nu_2$. Thus we are done. □

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